# Group-theoretical derivation of quadratic elastic invariants of two-dimensional quasicrystals of rank five and rank seven



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# Group-theoretical derivation of quadratic elastic invariants of two-dimensional quasicrystals of rank five and rank seven

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**Abstract.** Transformation matrices of phonon and phason strains under symmetry groups of two-dimensional (2D) quasicrystals (QCs) which are three-dimensional solids periodically stacked by aperiodic planes have been derived by using group representation theory. Quadratic invariants have been calculated for all 2D QCs of rank 5 and rank 7.

## 1. Introduction

In the past few decades, quasicrystals (QCs) have been studied extensively and thoroughly in many areas, one of which is symmetries and elastic properties. The linear elasticity behaviour of two-dimensional (2D) QCs of rank 5 [1–3] and rank 7 [4–6] have been discussed. In order to investigate the elastic behaviour the first step is to determine how many quadratic invariants there are and what they are.

As is well known, the invariants of a physical-property tensor in a certain structure are determined by the point-group symmetry which the structure possesses. It follows that the invariants of all kinds of physical-property tensor can be obtained with group representation theory. For periodic structures, systematic results have already been given (see, e.g., [8]).

A QC structure in a $d$-dimensional subspace (the physical space) $V_E$ can be obtained by intersecting a lattice-periodic structure in an $n$-dimensional embedding space $V$ with this subspace, where the space $V$ is the direct sum of $V_E$ and $V_I$, and $V_I$ is the orthogonal complement of the physical subspace. Recently, Janssen [4] gave a clear theoretical explanation for quasiperiodic structures and pointed out that such structures may have either crystallographic or non-crystallographic point-group symmetries. With this consideration, Hu *et al* [6, 9] have derived all the possible point groups of 2D QCs of rank 5 and rank 7. In addition, we have also proposed a method for determining the number of independent physical constants (i.e. the number of invariants) of QCs. In this paper we would like to give an alternative method which makes it easier to obtain the quadratic forms of strain tensors.

This method is demonstrated in section 2. The explicit quadratic forms are given with onefold, twofold, threefold, fourfold, fivefold, sixfold, sevenfold, eightfold, ninefold, tenfold, twelvefold, fourteenfold and eighteenfold rotational symmetries in section 3. Some remarks are made in section 4.

## 2. Fundamental theory

### 2.1. The basic transformation matrices of vectors

As in the previous paper [10], $\hat{A}$ and $\hat{A}'$ are the coordinate transformation matrices of the physical subspace and complementary subspace, respectively. For a 2D QC of rank 5, the physical subspace is three dimensional (3D), and the complementary subspace is 2D. If the $N$-fold axis is along $Z$ direction, the matrices $\hat{A}$ and $\hat{A}'$ are

$$\hat{A} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \hat{A}' = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \tag{1}$$

where $\alpha = 2\pi/N$, $\beta = p\alpha$, $1 \leqslant p < N$, $p$ and $N$ are relative prime. For the 2D QC of rank 7, such as the QCs with sevenfold, ninefold, fourteenfold or eighteenfold symmetry, besides $\hat{A}$ and $\hat{A}'$, there is another coordinate transformation matrix $\hat{A}''$ of complementary space with rotation angle $\gamma = q\alpha$, $p \neq q \neq 1$, and $p$ and $N$ are relative prime. So are $q$ and $N$. The numbers $p$ and $q$ are determined by the symmetry obeyed by the QC [4].

### 2.2. Transformation matrices of strains

In QCs there are two types of strain: phonon strain and phason strain. In general, the representation of a vector in physical subspace for a 2D QC can be divided into two parts: $\Gamma_z$ (one dimensional (1D) representation) and $\Gamma^{\parallel}_{x-y}$ (2D representation with a rotation angle $\alpha$). That in complementary subspace is another 2D representation $\Gamma^{\perp}_{x-y}$ with a rotation angle $\beta$. For the 2D QC with a crystallographic symmetry, $\Gamma^{\perp}_{x-y} = \Gamma^{\parallel}_{x-y}$; otherwise, $\Gamma^{\perp}_{x-y}$ is not equivalent to $\Gamma^{\parallel}_{x-y}$. Let us consider the point groups $C_n$, generated by a proper rotation, so that $\Gamma_z = \Gamma_1$, the identity representation. The mathematical treatment can be easily extended to the other point groups which include inversion $i$ ($x \to -x$, $y \to -y$, $z \to -z$), or horizontal mirror reflection $m_h$ ($x \to x$, $y \to y$, $z \to -z$), or vertical mirror reflection $m_v$ ($x \to x$, $y \to -y$, $z \to z$ or $x \to -x$ $y \to y$, $z \to z$), or horizontal twofold rotation $2_h$ ($x \to x$, $y \to -y$, $z \to -z$ or $x \to -x$, $y \to y$, $z \to -z$).

For the phonon strain field, the six components of $E_{ij}$ transform under

$$((\Gamma_1 + \Gamma^{\parallel}_{x-y}) \otimes (\Gamma_1 + \Gamma^{\parallel}_{x-y}))^S = 2\Gamma_1 + \Gamma^{\parallel}_{x-y} + \Gamma_{II} \tag{2}$$

where $E_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$, the superscript $S$ means the symmetrical part, $E_{11} + E_{22}$ and $E_{33}$ span the two identity representations, and $(E_{13}, E_{23})$ *and* $(E_{11} - E_{22}, 2E_{12})$ span the two 2D representations $\Gamma^{\parallel}_{x-y}$ (with rotation angle $\alpha$) and $\Gamma_{II}$ (with rotation angle $2\alpha$), respectively. The explicit expressions are as follows:

$$(E_{11} + E_{22})' = E_{11} + E_{22}$$
$$E'_{33} = E_{33}$$
$$\begin{bmatrix} E_{13} \\ E_{23} \end{bmatrix}' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} E_{13} \\ E_{23} \end{bmatrix} = \hat{M}(\alpha) \begin{bmatrix} E_{13} \\ E_{23} \end{bmatrix}$$
$$\begin{bmatrix} E_{11} - E_{22} \\ 2E_{12} \end{bmatrix}' = \begin{bmatrix} \cos(2\alpha) & -\sin(2\alpha) \\ \sin(2\alpha) & \cos(2\alpha) \end{bmatrix} \begin{bmatrix} E_{11} - E_{22} \\ 2E_{12} \end{bmatrix} = \hat{M}(2\alpha) \begin{bmatrix} E_{11} - E_{22} \\ 2E_{12} \end{bmatrix} \tag{3}$$

where the terms in square brackets are related to the old coordinate system, and those in primed square brackets to the new coordinate system.

The phason strain $\partial_j w_i$ transforms under

$$(\Gamma_1 + \Gamma^{\parallel}_{x-y}) \otimes \Gamma^{\perp}_{x-y} = \Gamma^{\perp}_{x-y} + \Gamma'_{II} + \Gamma''_{II}. \tag{4}$$

It follows that $(\partial_3 w_1, \partial_3 w_2)$, $(\partial_1 w_1 - \partial_2 w_2, \partial_1 w_1 + \partial_2 w_2)$, $(\partial_1 w_1 + \partial_2 w_2, \partial_1 w_2 - \partial_2 w_1)$ span the representation $\Gamma^{\perp}_{x-y}$ (with rotation angle $\beta$), $\Gamma'_{II}$ (with $(\beta+\alpha)$), and $\Gamma''_{II}$ (with $(\beta-\alpha)$), respectively, i.e.

$$\begin{aligned}
\begin{bmatrix}\partial_3 w_1\\ \partial_3 w_2\end{bmatrix}' &= \begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}\begin{bmatrix}\partial_3 w_1\\ \partial_3 w_2\end{bmatrix}\\
\begin{bmatrix}\partial_1 w_1 - \partial_2 w_2\\ \partial_1 w_2 + \partial_2 w_1\end{bmatrix}' &= \begin{bmatrix}\cos(\beta+\alpha) & -\sin(\beta+\alpha)\\ \sin(\beta+\alpha) & \cos(\beta+\alpha)\end{bmatrix}\begin{bmatrix}\partial_1 w_1 - \partial_2 w_2\\ \partial_1 w_2 + \partial_2 w_1\end{bmatrix}\\
\begin{bmatrix}\partial_1 w_1 + \partial_2 w_2\\ \partial_1 w_2 - \partial_2 w_1\end{bmatrix}' &= \begin{bmatrix}\cos(\beta-\alpha) & -\sin(\beta-\alpha)\\ \sin(\beta-\alpha) & \cos(\beta-\alpha)\end{bmatrix}\begin{bmatrix}\partial_1 w_1 + \partial_2 w_2\\ \partial_1 w_2 - \partial_2 w_1\end{bmatrix}.
\end{aligned} \tag{5}$$

For the 2D QC of rank 7, there is another type of phason strain $\partial_j v_i$; substituting $\beta$ by $\gamma$ in equations (5), one can obtain similar results for $\partial_j v_i$.

### *2.3. Possible quadratic invariants of phonon strain, phason strain and coupling between phonon strain and phason strain in two-dimensional quasicrystals*

In QCs, there are three types of quadratic invariant contributing to linear elastic energy: phonon strain $\sum E_{ij}E_{kl}$, phason strain $\sum \partial_j w_i \partial_l w_k$ and coupling between phonon strain and phason strain $\sum E_{ij}\partial_l w_k$. In the following, we shall discuss these three types of quadratic invariant.

*2.3.1. Quadratic invariants of phonon strain.* For conventional crystals, the linear elastic energy is determined only by this term, and only one rotational angle $\alpha$ is associated with this type of invariant. In QCs, this term is similar to that of crystals.

For the QC of rank 5 or rank 7, only onefold, twofold, threefold, fourfold, fivefold, sixfold, sevenfold, eightfold, ninefold, tenfold, twelvefold, fourteenfold or eighteenfold symmetry is allowable; the rotation angle $\alpha = 2\pi/N$.

In equation (3), there are two linear invariants $E_{11}+E_{22}$ and $E_{33}$, giving three quadratic invariants $(E_{11}+E_{22})^2$, $E_{33}^2$ and $(E_{11}+E_{22})E_{33}$.

(i) If $\alpha = 2\pi$ $(N = 1)$, the remaining four symmetric components: $E_{13}$, $E_{23}$, $E_{11}-E_{22}$ *and* $E_{12}$ are also first-order linear invariants; so there are 21 quadratic invariants as in triclinic crystals.

(ii) If $\alpha = \pi$ $(N = 2)$, the remaining four components transform according to

$$E'_{13} = -E_{13} \qquad E'_{23} = -E_{23} \qquad (E_{11}-E_{22})' = E_{11}-E_{22} \qquad E'_{12} = E_{12}. \tag{6}$$

It follows that, among six phonon strains $E_{ij}$, four transform under the identity representation, and two transform under the 1D antisymmetric representation, producing 13 quadratic invariants. They are $E_{13}^2$, $E_{23}^2$, $E_{13}E_{23}$ and the products of the four linear invariants.

(iii) If $\alpha = \pi/2$ $(N = 4)$, the components $(E_{11}-E_{22}, 2E_{12})$ transform according to

$$\begin{aligned}
(E_{11}-E_{22})' &= -(E_{11}-E_{22})\\
E'_{12} &= -E_{12}
\end{aligned} \tag{7}$$

giving three quadratic invariants $(E_{11}-E_{22})^2$, $E_{12}^2$ and $(E_{11}-E_{22})E_{12}$. Meanwhile the components $E_{13}$ and $E_{23}$ give rise to one quadratic form $E_{13}^2+E_{23}^2$. There are seven quadratic invariants all together.

(iv) If $N$ is equal to the other integers, neither $\Gamma^{\parallel}_{x-y}$ nor $\Gamma_{II}$ can be decomposed any longer; in this case the dot products of the pairs $(E_{13}, E_{23})$ and $(E_{11}-E_{22}, 2E_{12})$ can be expressed as follows:

$$
\begin{aligned}
[E_{13}, E_{23}]'\begin{bmatrix}E_{13}\\E_{23}\end{bmatrix}' &= [E_{13}, E_{23}]\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}E_{12}\\E_{23}\end{bmatrix}\\
&= [E_{13}, E_{23}]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}E_{13}\\E_{23}\end{bmatrix}\\
[E_{11}-E_{22}, 2E_{12}]'\begin{bmatrix}E_{11}-E_{22}\\2E_{12}\end{bmatrix}' &= [E_{11}-E_{22}, 2E_{12}]\begin{bmatrix}\cos(2\alpha) & \sin(2\alpha)\\ -\sin(2\alpha) & \cos(2\alpha)\end{bmatrix}\\
&\quad\times\begin{bmatrix}\cos(2\alpha) & -\sin(2\alpha)\\ \sin(2\alpha) & \cos(2\alpha)\end{bmatrix}\begin{bmatrix}E_{11}-E_{22}\\2E_{12}\end{bmatrix}\\
&= [E_{11}-E_{22}, 2E_{12}]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}E_{11}-E_{22}\\2E_{12}\end{bmatrix}\\
[E_{13}, E_{23}]'\begin{bmatrix}E_{11}-E_{22}\\2E_{12}\end{bmatrix}' &= [E_{13}, E_{23}]\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}E_{11}-E_{22}\\2E_{12}\end{bmatrix}\\
&= [E_{13}, E_{23}]\hat{M}(\alpha)\begin{bmatrix}E_{11}-E_{22}\\2E_{12}\end{bmatrix}\\
[E_{13}, E_{23}]'\begin{bmatrix}2E_{12}\\E_{11}-E_{22}\end{bmatrix} &= [E_{13}, E_{23}]\begin{bmatrix}\cos(3\alpha) & \sin(3\alpha)\\ -\sin(3\alpha) & \cos(3\alpha)\end{bmatrix}\begin{bmatrix}2E_{12}\\E_{11}-E_{22}\end{bmatrix}\\
&= [E_{13}, E_{23}]\hat{M}(-3\alpha)\begin{bmatrix}2E_{12}\\E_{11}-E_{22}\end{bmatrix}.
\end{aligned}
\tag{8}
$$

Obviously, the first two products in equation (8) are invariants. For the last two expressions, if and only if $\theta = m2\pi$ with $m$ being integer, $\hat{M}(\theta)$ is a unit matrix; hence the corresponding dot proudct is an invariant. Therefore there are least five quadratic invariants (essential phonon invariants), i.e. $(E_{11}+E_{22})^2$, $E_{33}^2$, $(E_{11}+E_{22})E_{33}$, $E_{13}^2+E_{23}^2$ and $(E_{11}-E_{22})^2+4E_{12}^2$ for any 2D QC.

*2.3.2. Quadratic invariants of phason strain.* (i) $N = 1, 2, 3, 4$, or 6: this is the case of QCs with crystallographic symmetries and of rank 5, in this case $\beta = \alpha$. By comparing equations (3) and (5), one can find that $\begin{bmatrix}\partial_3 w_1\\ \partial_3 w_2\end{bmatrix}$ *and* $\begin{bmatrix}E_{13}\\E_{23}\end{bmatrix}$, $\begin{bmatrix}\partial_1 w_1-\partial_2 w_2\\ \partial_1 w_2+\partial_2 w_1\end{bmatrix}$ *and* $\begin{bmatrix}E_{11}-E_{22}\\2E_{12}\end{bmatrix}$, $\partial_1 w_1+\partial_2 w_2$ *and* $E_{11}+E_{22}$, and $\partial_1 w_2-\partial_2 w_1$ *and* $E_{33}$ take the same transformation matrices, respectively. So, with the corresponding substitutions, the quadratic invariants of phason strain for this case take similar forms as that of phonon strain discussed above.

(ii) $N = 5, 8, 10$, or 12: this is the case of QCs with non-crystallographic symmetries and of rank 5. In this case $\beta = p\alpha$, $p = 3, 3, 3, 5$, respectively.

In particular, when $N = 8, 12$, $\beta+\alpha = \pi$, then $\Gamma'_{II}$ in equaiton (4) can be decomposed into two 1D antisymmetric representations, which give three quadratic invariants $(\partial_1 w_1 - \partial_2 w_2)^2$, $(\partial_1 w_2+\partial_2 w_1)^2$ and $(\partial_1 w_1-\partial_2 w_2)(\partial_1 w_2+\partial_2 w_1)$. From equation (5), three invariants $(\partial_3 w_1)^2+(\partial_3 w_2)^2$, $(\partial_1 w_1-\partial_2 w_2)^2+(\partial_1 w_2+\partial_2 w_1)^2$ and $(\partial_1 w_1+\partial_2 w_2)^2+(\partial_1 w_2-\partial_2 w_1)^2$ always exist in any case. These three invariants can be called essential phason invariants. The other invariants can be determined by the following dot products with the transformation matrices $\hat{M}(\theta)$:

| dot product | transformation matrix $\hat{M}(\theta)$ |
| --- | --- |
| $[\partial_3 w_1, \partial_3 w_2]\begin{bmatrix}\partial_1 w_1 - \partial_2 w_2\\ \partial_1 w_2 + \partial_2 w_1\end{bmatrix}$ | $\hat{M}(\alpha)$ |
| $[\partial_3 w_1, \partial_3 w_2]\begin{bmatrix}\partial_1 w_2 + \partial_2 w_1\\ \partial_1 w_1 - \partial_2 w_2\end{bmatrix}$ | $\hat{M}(-\alpha - 2\beta)$ |
| $[\partial_3 w_1, \partial_3 w_2]\begin{bmatrix}\partial_1 w_1 + \partial_2 w_2\\ \partial_1 w_2 - \partial_2 w_1\end{bmatrix}$ | $\hat{M}(-\alpha)$ |
| $[\partial_3 w_1, \partial_3 w_2]\begin{bmatrix}\partial_1 w_2 - \partial_2 w_1\\ \partial_1 w_1 + \partial_2 w_2\end{bmatrix}$ | $\hat{M}(\alpha - 2\beta)$ |
| $[\partial_1 w_1 - \partial_2 w_2, \partial_1 w_2 + \partial_2 w_1]\begin{bmatrix}\partial_1 w_1 + \partial_2 w_2\\ \partial_1 w_2 - \partial_2 w_1\end{bmatrix}$ | $\hat{M}(-2\alpha)$ |
| $[\partial_1 w_1 - \partial_2 w_2, \partial_1 w_2 + \partial_2 w_1]\begin{bmatrix}\partial_1 w_2 - \partial_2 w_1\\ \partial_1 w_1 + \partial_2 w_2\end{bmatrix}$ | $\hat{M}(-2\beta)$. (9) |

(iii) $N = 7, 9, 14$, or 18: this is the case of QCs of rank 7. There are two types of phason strain, namely $\partial_j w_i$ *and* $\partial_j v_i$ *with* $\beta = p\alpha$ *and* $\gamma = q\alpha$, for the folflowing $p$- and $q$-values: $p = 5, q = 3$; $p = 2, q = 4$; $p = 3, q = 5$; $p = 5, q = 7$. So, there are three types of quadratic invariant of phason strain, two self-products ($\partial_j w_i \partial_l w_k$ and $\partial_j v_i \partial_l v_k$) and one cross-term ($\partial_j w_i \partial_l v_k$). The quadratic invariants due to self-products can be obtained in the same manner as in (ii). The possible dot products used to construct the invariants due to the cross-term are as follows:

| dot product | transformation matrix $\hat{M}(\theta)$ |
| --- | --- |
| $[\partial_3 w_1, \partial_3 w_2]\begin{bmatrix}\partial_3 v_1\\ \partial_3 v_2\end{bmatrix}$ | $\hat{M}(\gamma - \beta)$ |
| $[\partial_3 w_1, \partial_3 w_2]\begin{bmatrix}\partial_3 v_2\\ \partial_3 v_1\end{bmatrix}$ | $\hat{M}(-\gamma - \beta)$ |
| $[\partial_1 w_1 - \partial_2 w_2, \partial_1 w_2 + \partial_2 w_1]\begin{bmatrix}\partial_1 v_1 - \partial_2 v_2\\ \partial_1 v_2 + \partial_2 v_1\end{bmatrix}$ | $\hat{M}(\gamma - \beta)$ |
| $[\partial_1 w_1 - \partial_2 w_2, \partial_1 w_2 + \partial_2 w_1]\begin{bmatrix}\partial_1 v_2 + \partial_2 v_1\\ \partial_1 v_1 - \partial_2 v_2\end{bmatrix}$ | $\hat{M}(-2\alpha - \beta - \gamma)$ |
| $[\partial_1 w_1 + \partial_2 w_2, \partial_1 w_2 - \partial_2 w_1]\begin{bmatrix}\partial_1 v_1 + \partial_2 v_2\\ \partial_1 v_2 - \partial_2 v_1\end{bmatrix}$ | $\hat{M}(\gamma - \beta)$ |
| $[\partial_1 w_1 + \partial_2 w_2, \partial_1 w_2 - \partial_2 w_1]\begin{bmatrix}\partial_1 v_2 - \partial_2 v_1\\ \partial_1 v_1 + \partial_2 v_2\end{bmatrix}$ | $\hat{M}(2\alpha - \gamma - \beta)$ |
| $[\partial_3 w_1, \partial_3 w_2]\begin{bmatrix}\partial_1 v_1 - \partial_2 v_2\\ \partial_1 v_2 + \partial_2 v_1\end{bmatrix}$ | $\hat{M}(\gamma - \beta + \alpha)$ |
| $[\partial_3 w_1, \partial_3 w_2]\begin{bmatrix}\partial_1 v_2 + \partial_2 v_1\\ \partial_1 v_1 - \partial_2 v_2\end{bmatrix}$ | $\hat{M}(-\gamma - \beta - \alpha)$ |
| $[\partial_3 w_1, \partial_3 w_2]\begin{bmatrix}\partial_1 v_1 + \partial_2 v_2\\ \partial_1 v_2 - \partial_2 v_1\end{bmatrix}$ | $\hat{M}(\gamma - \beta - \alpha)$ |
| $[\partial_3 w_1, \partial_3 w_2]\begin{bmatrix}\partial_1 v_2 - \partial_2 v_1\\ \partial_1 v_1 + \partial_2 v_2\end{bmatrix}$ | $\hat{M}(\alpha - \gamma - \beta)$ |

$$[\partial_1 w_1 - \partial_2 w_2, \partial_1 w_2 + \partial_2 w_1]\begin{bmatrix}\partial_1 v_1 + \partial_2 v_2 \\ \partial_1 v_2 - \partial_2 v_1\end{bmatrix} \qquad \hat{M}(\gamma - \beta - 2\alpha)$$

$$[\partial_1 w_1 - \partial_2 w_2, \partial_1 w_2 + \partial_2 w_1]\begin{bmatrix}\partial_1 v_2 - \partial_2 v_1 \\ \partial_1 v_1 + \partial_2 v_2\end{bmatrix} \qquad \hat{M}(-\beta - \gamma)$$

$$[\partial_1 w_1 + \partial_2 w_2, \partial_1 w_2 - \partial_2 w_1]\begin{bmatrix}\partial_3 v_1 \\ \partial_3 v_2\end{bmatrix} \qquad \hat{M}(\gamma - \beta + \alpha)$$

$$[\partial_1 w_1 + \partial_2 w_2, \partial_1 w_2 - \partial_2 w_1]\begin{bmatrix}\partial_3 v_2 \\ \partial_3 v_1\end{bmatrix} \qquad \hat{M}(\alpha - \beta - \gamma)$$

$$[\partial_1 w_1 - \partial_2 w_2, \partial_1 w_2 + \partial_2 w_1]\begin{bmatrix}\partial_3 v_2 \\ \partial_3 v_1\end{bmatrix} \qquad \hat{M}(-\alpha - \beta - \gamma)$$

$$[\partial_1 w_1 - \partial_2 w_2, \partial_1 w_2 + \partial_2 w_1]\begin{bmatrix}\partial_3 v_1 \\ \partial_3 v_2\end{bmatrix} \qquad \hat{M}(\gamma - \beta - \alpha)$$

$$[\partial_1 w_1 + \partial_2 w_2, \partial_1 w_2 - \partial_2 w_1]\begin{bmatrix}\partial_1 v_1 - \partial_2 v_2 \\ \partial_1 v_2 + \partial_2 v_1\end{bmatrix} \qquad \hat{M}(\gamma - \beta + 2\alpha)$$

$$[\partial_1 w_1 + \partial_2 w_2, \partial_1 w_2 - \partial_2 w_1]\begin{bmatrix}\partial_1 v_2 + \partial_2 v_1 \\ \partial_1 v_1 - \partial_2 v_2\end{bmatrix} \qquad \hat{M}(-\beta - \gamma). \qquad (10)$$

*2.3.3. Coupling between phonon strain and phason strain.* If there are common representations in $E_{ij}$ and $\partial_j w_i$ (or $\partial_j v_i$), there must exist coupling invariants between phonon strain and phason strain.

(i) For 2D QCs with crystallographic symmetries, $E_{ij}$ and $\partial_j w_i$ transform under the same representation. The coupling invariants between phonon strain and phason strain can be easily obtained by the dot product between the basis vector of the 1D rational representation in $E_{ij}$ and that of the same representation in $\partial_l w_k$ and between the basis vector of the 2D rational representation in $E_{ij}$ and that of the same representation in $\partial_l w_k$.

(ii) For 2D QCs with non-crystallographic symmetries and of rank 5, all the possible quadratic invariants can be obtained by the dot products betweens $[E_{13}, E_{23}]$, $[E_{11} - E_{22}, 2E_{12}]$ and $[\partial_3 w_1, \partial_3 w_2]^{\mathrm{T}}$, $[\partial_1 w_1 - \partial_2 w_2, \partial_1 w_2 + \partial_2 w_1]^{\mathrm{T}}$, $[\partial_1 w_1 - \partial_2 w_2, \partial_1 w_2 + \partial_2 w_1]^{\mathrm{T}}$:

| dot product | transformation matrix $\hat{M}(\theta)$ |
|---|---|
| $[E_{13}, E_{23}]\begin{bmatrix}\partial_3 w_1 \\ \partial_3 w_2\end{bmatrix}$ | $\hat{M}(\beta - \alpha)$ |
| $[E_{13}, E_{23}]\begin{bmatrix}\partial_3 w_2 \\ \partial_3 w_1\end{bmatrix}$ | $\hat{M}(-\beta - \alpha)$ |
| $[E_{13}, E_{23}]\begin{bmatrix}\partial_1 w_1 - \partial_2 w_2 \\ \partial_1 w_2 + \partial_2 w_1\end{bmatrix}$ | $\hat{M}(\beta)$ |
| $[E_{13}, E_{23}]\begin{bmatrix}\partial_1 w_2 + \partial_2 w_1 \\ \partial_1 w_1 - \partial_2 w_2\end{bmatrix}$ | $\hat{M}(-\beta - 2\alpha)$ |
| $[E_{13}, E_{23}]\begin{bmatrix}\partial_1 w_1 + \partial_2 w_2 \\ \partial_1 w_2 - \partial_2 w_1\end{bmatrix}$ | $\hat{M}(\beta - 2\alpha)$ |
| $[E_{13}, E_{23}]\begin{bmatrix}\partial_1 w_2 - \partial_2 w_1 \\ \partial_1 w_1 + \partial_2 w_2\end{bmatrix}$ | $\hat{M}(-\beta)$ |
| $[E_{11} - E_{22}, 2E_{12}]\begin{bmatrix}\partial_3 w_1 \\ \partial_3 w_2\end{bmatrix}$ | $\hat{M}(\beta - 2\alpha)$ |

$$[E_{11}-E_{22},2E_{12}]\begin{bmatrix}\partial_3 w_2\\ \partial_3 w_1\end{bmatrix} \qquad \hat{M}(-\beta-2\alpha)$$

$$[E_{11}-E_{22},2E_{12}]\begin{bmatrix}\partial_1 w_1-\partial_2 w_2\\ \partial_1 w_2+\partial_2 w_1\end{bmatrix} \qquad \hat{M}(\beta-\alpha)$$

$$[E_{11}-E_{22},2E_{12}]\begin{bmatrix}\partial_1 w_2+\partial_2 w_1\\ \partial_1 w_1-\partial_2 w_2\end{bmatrix} \qquad \hat{M}(-\beta-3\alpha)$$

$$[E_{11}-E_{22},2E_{12}]\begin{bmatrix}\partial_1 w_1+\partial_2 w_2\\ \partial_1 w_2-\partial_2 w_1\end{bmatrix} \qquad \hat{M}(\beta-3\alpha)$$

$$[E_{11}-E_{22},2E_{12}]\begin{bmatrix}\partial_1 w_2-\partial_2 w_1\\ \partial_1 w_1+\partial_2 w_2\end{bmatrix} \qquad \hat{M}(-\beta-\alpha). \tag{11}$$

(iii) For 2D QCs of rank 7, the possible coupling invariants between phonon strain and phason strain take the form of either $E_{ij}\partial_l w_k$ or $E_{ij}\partial_l v_k$. Substituting $\beta$ in (11) by $p\alpha$ and $q\alpha$, we obtain all possible invariants.

It should be noted that, in (8)–(11), $[A,B]\begin{bmatrix}C\\ D\end{bmatrix}$ and $[A,B]\begin{bmatrix}D\\ -C\end{bmatrix}$ transform under the same representation; hence, if $AC+BD$ is an invariant, $AD-BC$ is also an invariant.

## 3. Some examples (application)

In section 2, we have given all the possible quadratic elastic invariants for 2D QCs. In following, we shall discuss the cases $N=3$, 5 and 7, as examples of three types of 2D QC. A similar procedure can be used for the other cases.

### *3.1. $N=3$: the case for a two-dimensional quasicrystal with crystallographic symmetry*

In (8), $[E_{13},E_{23}]\begin{bmatrix}2E_{12}\\ E_{11}-E_{22}\end{bmatrix}$ transforms under the identity representation due to $3\alpha=2\pi$; thus, besides the five essential phonon invariants, another two invariants, namely $2E_{13}E_{12}+E_{23}(E_{11}-E_{22})$ and $E_{13}(E_{11}-E_{22})-2E_{23}E_{12}$ can be obtained.

Following the discussion in section 2.3.2, the seven similar quadratic invariants of phason strain can be written as $(\partial_1 w_1+\partial_2 w_2)^2$, $(\partial_1 w_2-\partial_2 w_1)^2$, $(\partial_1 w_1+\partial_2 w_2)(\partial_1 w_2-\partial_2 w_1)$, $(\partial_3 w_1)^2+(\partial_3 w_2)^2$, $(\partial_1 w_1-\partial_2 w_2)^2+(\partial_1 w_2+\partial_2 w_1)^2$, $\partial_3 w_1(\partial_1 w_2+\partial_2 w_1)+\partial_3 w_2(\partial_1 w_1-\partial_2 w_2)$ and $\partial_3 w_1(\partial_1 w_1-\partial_2 w_2)-\partial_3 w_2(\partial_1 w_2+\partial_2 w_1)$.

In (11), $\beta-\alpha=0$, $\beta+2\alpha=2\pi$; thus, $[E_{13},E_{23}]\begin{bmatrix}\partial_3 w_1\\ \partial_3 w_2\end{bmatrix}$, $[E_{13},E_{23}]\begin{bmatrix}\partial_1 w_2+\partial_2 w_1\\ \partial_1 w_1-\partial_2 w_2\end{bmatrix}$, $[E_{11}-E_{22},2E_{12}]\begin{bmatrix}\partial_3 w_2\\ \partial_3 w_1\end{bmatrix}$, and $[E_{11}-E_{22},2E_{12}]\begin{bmatrix}\partial_1 w_1-\partial_2 w_2\\ \partial_1 w_2+\partial_2 w_1\end{bmatrix}$ transform under the identity representation, respectively, which contribute to eight quadratic invariants. Combining with the four invariants obtained by dot product between $E_{11}+E_{22}$, $E_{33}$ and $\partial_1 w_1+\partial_2 w_2$, $\partial_1 w_2-\partial_2 w_1$, a total of 12 quadratic invariants can be obtained.

### *3.2. $N=5$: the case of a two-dimensional quasicrystal with non-crystallographic symmetry and of rank 5*

In equation (8), neither $\alpha$ nor $3\alpha$ equals 0 or $2\pi$; so there are only five essential phonon invariants, and this holds for any $N\geqslant 5$ case. From the consideration of the phonon field, all the cases with $N\geqslant 5$ can be called transverse isotopic.

**Table 1.** Quadratic elastic invariants of 2D QCs. In this table, $E_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$, $W_{ij} = \partial j w_i$ and $V_{ij} = \partial_j v_i$.

| $N$ in $N$-fold | Rank of QC | Rotation angles | Special relations of rotation angles | Numbers of quadratic invariants | Quadratic invariants |
|---|---|---|---|---|---|
| 1 | 5 | $\alpha = 2\pi$<br>$\beta = \alpha$ | — | $n_C = 21(13)$<br>$n_K = 21(12)$<br>$n_R = 36(18)$ | Dot product of<br>$A = \{E_{11}, E_{22}, E_{33}, E_{23}, W_{11}, W_{22}, W_{23}\}$<br>and $B = \{E_{12}, E_{13}, W_{12}, W_{13}, W_{21}\}$;<br>$\langle$dot product between A and B$\rangle$ |
| 2 | 5 | $\alpha = \pi$<br>$\beta = \alpha$ | — | $n_C = 13(9)$<br>$n_K = 13(8)$<br>$n_R = 20(10)$ | Dot product of<br>$A = \{E_{11}, E_{22}, E_{33}, W_{11}, W_{22}\}$<br>and $B = \{E_{12}, W_{12}, W_{21}\}$;<br>$\langle$dot product between A and B$\rangle$<br>$E_{13}^2, E_{23}^2, W_{13}^2, W_{23}^2, E_{13}W_{13}, E_{23}W_{23}$,<br>$\langle E_{13}E_{23}, W_{13}W_{23}, E_{23}W_{13}, E_{13}W_{23}\rangle$ |
| 3 | 5 | $\alpha = \frac{2}{3}\pi$<br>$\beta = \alpha$ | $3\alpha = 2\pi$ | $n_C = 7(6)$<br>$n_K = 7(5)$<br>$n_R = 12(6)$ | Dot product of<br>$A = \{E_{33}, E_{11} + E_{22}, W_{11} + W_{22}\}$<br>and $B = \{W_{21} - W_{12}\}$;<br>$\langle$dot product between A and B$\rangle$<br>$E_{13}^2 + E_{23}^2, 4E_{12}^2 + (E_{11} - E_{22})^2$,<br>$2E_{13}E_{12} + E_{23}(E_{11} - E_{22})$,<br>$\langle E_{13}(E_{11} - E_{22}) - 2E_{23}E_{12}\rangle$,<br>$W_{13}^2 + W_{23}^2, (W_{12} + W_{21})^2 + (W_{11} - W_{22})^2$,<br>$(W_{12} + W_{21})W_{13} + (W_{11} - W_{22})W_{23}$,<br>$\langle W_{13}(W_{11} - W_{22}) - (W_{12} + W_{21})W_{23}\rangle$,<br>$2(W_{12} + W_{21})E_{12} + (W_{11} - W_{22})(E_{11} - E_{22})$,<br>$\langle 2(W_{11} - W_{22})E_{12} - (W_{12} + W_{21})(E_{11} - E_{22})\rangle$,<br>$E_{13}W_{13} + E_{23}W_{23}, \langle E_{13}W_{23} - E_{23}W_{13}\rangle$,<br>$(W_{12} + W_{21})E_{13} + (W_{11} - W_{22})E_{23}$,<br>$\langle (W_{11} - W_{22})E_{13} - (W_{12} + W_{21})E_{23}\rangle$,<br>$2E_{12}W_{13} + (E_{11} - E_{22})E_{23}$,<br>$\langle 2E_{12}W_{23} - (E_{11} - E_{22})W_{13}\rangle$ |

Table 1. (Continued)

| $N$ in $N$-fold | Rank of QC | Rotation angles | Special relations of rotation angles | Numbers of quadratic invariants | Quadratic invariants |
|---|---|---|---|---|---|
| 4 | 5 | $\alpha = \frac{1}{2}\pi$<br>$\beta = \alpha$ | — | $n_C = 7(6)$<br>$n_k = 7(5)$<br>$n_R = 10(5)$ | Dot product of<br>$A = \{E_{33}, E_{11} + E_{22}, W_{11} + W_{22}\}$ and $B = \{W_{21} - W_{12}\}$;<br>(dot product between A and B)<br>$E_{12}^2, (E_{11} - E_{22})^2, E_{13}^2 + E_{23}^2$,<br>$\langle E_{12}(E_{11} - E_{22})\rangle, (W_{11} - W_{22})^2$,<br>$W_{13}^2 + W_{23}^2, (W_{12} + W_{21})^2, \langle (W_{11} - W_{22})(W_{12} + W_{21})\rangle$,<br>$(W_{12} + W_{21})E_{12}, (W_{11} - W_{22})(E_{11} - E_{22})$,<br>$E_{13}W_{13} + E_{23}W_{23}, \langle E_{13}W_{23} - E_{23}W_{13}\rangle$,<br>$\langle E_{12}(W_{11} - W_{22})\rangle, \langle (E_{11} - E_{22})(W_{12} + W_{21})\rangle$ |
| 5 | 5 | $\alpha = \frac{2}{5}\pi$<br>$\beta = 3\alpha$ | $\alpha - \beta = -2\pi$<br>$\beta + 2\alpha = \pi$<br>$\beta - 3\alpha = 0$ | $n_C = 5(5)$<br>$n_K = 5(4)$<br>$n_R = 6(3)$ | Essential phonon invariants $E_{ij}E_{kl}$;<br>essential phason invariants $W_{ij}W_{kl}$:<br>$(W_{21} - W_{12})W_{13} + (W_{11} + W_{22})W_{23}$,<br>$\langle (W_{11} + W_{22})W_{13} - (W_{21} - W_{12})W_{23}\rangle$,<br>$(W_{11} - W_{22})E_{23} + (W_{12} + W_{21})E_{13}$,<br>$\langle E_{13}(W_{11} - W_{22}) - (W_{21} + W_{12})E_{23}\rangle$,<br>$(E_{11} - E_{22})W_{23} + 2E_{12}W_{13}$,<br>$\langle (E_{11} - E_{22})W_{13} - 2E_{12}W_{23}\rangle$,<br>$2(W_{21} - W_{12})E_{12} + (W_{11} + W_{22})(E_{11} - E_{22})$,<br>$\langle (E_{11} - E_{22})(W_{21} - W_{12}) - 2E_{12}(W_{11} + W_{22})\rangle$ |
| 6 | 5 | $\alpha = \frac{1}{3}\pi$<br>$\beta = \alpha$ | — | $n_C = 5(5)$<br>$n_K = 5(4)$<br>$n_R = 8(4)$ | Dot product of<br>$A = \{E_{33}, E_{11} + E_{22}, W_{11} + W_{22}\}$ and $B = \{W_{21} - W_{12}\}$;<br>(dot product between A and B)<br>$E_{13}^2 + E_{23}^2, 4E_{12}^2 + (E_{11} - E_{22})^2, W_{13}^2 + W_{23}^2$,<br>$(W_{12} + W_{21})^2 + (W_{11} - W_{22})^2, E_{13}W_{13} + E_{23}W_{23}$,<br>$\langle E_{13}W_{23} - E_{23}W_{13}\rangle$,<br>$2(W_{12} + W_{21})E_{12} + (W_{11} - W_{22})(E_{11} - E_{22})$,<br>$\langle (E_{11} - E_{22})(W_{12} + W_{21}) - 2E_{12}(W_{11} - W_{22})\rangle$ |

Table 1. (Continued)

| $N$ in $N$-fold | Rank of QC | Rotation angles | Special relations of rotation angles | Numbers of quadratic invariants | Quadratic invariants |
|---|---|---|---|---|---|
| 7 | 7 | $\alpha = \frac{2}{7}\pi$<br>$\beta = 5\alpha$<br>$\gamma = 3\alpha$ | $\alpha + 2\gamma = 2\pi$<br>$2\alpha - \beta + \gamma = 0$<br>$\gamma + \beta - \alpha = 2\pi$<br>$\beta + 2\alpha = 2\pi$<br>$\gamma - 3\alpha = 0$ | $n_C = 5(5)$<br>$n_K = 14(10)$<br>$n_R = 6(3)$ | Essential phonon ($\vec{u}$) invariant $E_{ij}E_{kl}$;<br>essential phason ($\vec{w}$) invariant $W_{ij}W_{kl}$;<br>essential phason ($\vec{v}$) invariants $V_{ij}V_{kl}$:<br>$(V_{12} + V_{21})V_{13} + (V_{11} - V_{22})V_{23}$,<br>$\langle (V_{11} - V_{22})V_{13} - (V_{21} + V_{12})V_{23} \rangle$,<br>$(W_{11} + W_{22})(V_{11} - V_{22}) + (W_{21} - W_{12})(V_{21} + V_{12})$,<br>$\langle (W_{11} + W_{22})(V_{21} + V_{12}) - (W_{21} - W_{12})(V_{11} - V_{22}) \rangle$,<br>$(V_{21} - V_{12})W_{13} + (V_{11} + V_{22})W_{23}$,<br>$\langle (V_{11} + V_{22})W_{13} - (V_{21} - V_{12})W_{23} \rangle$,<br>$(W_{11} + W_{22})V_{23} + (W_{21} - W_{12})V_{13}$,<br>$\langle (W_{11} + W_{22})V_{13} - (W_{21} - W_{12})V_{23} \rangle$,<br>$(W_{12} + W_{21})E_{13} + (W_{11} - W_{22})E_{23}$,<br>$\langle E_{13}(W_{11} - W_{22}) - (W_{12} + W_{21})E_{23} \rangle$,<br>$(E_{11} - E_{22})W_{23} + 2E_{12}W_{13}$,<br>$\langle (E_{11} - E_{22})W_{13} - 2E_{12}W_{23} \rangle$,<br>$(E_{11} - E_{22})(V_{11} + V_{22}) + 2E_{12}(V_{21} - V_{12})$,<br>$\langle (E_{11} - E_{22})(V_{21} - V_{12}) - 2E_{12}(V_{11} + V_{22}) \rangle$ |
| 8 | 5 | $\alpha = \frac{1}{4}\pi$<br>$\beta = 3\alpha$ | $\beta + \alpha = \pi$<br>$\beta - 3\alpha = 0$ | $n_C = 5(5)$<br>$n_K = 5(4)$<br>$n_R = 2(1)$ | Essential phonon invariants $E_{ij}E_{kl}$;<br>essential phason invariants $W_{ij}W_{kl}$:<br>$(W_{11} - W_{22})^2$, $\langle (W_{11} - W_{22})(W_{21} + W_{12}) \rangle$,<br>$2(W_{21} - W_{12})E_{12} + (W_{11} + W_{22})(E_{11} - E_{22})$,<br>$\langle (E_{11} - E_{22})(W_{21} - W_{12}) - 2E_{12}(W_{11} + W_{22}) \rangle$ |

**Table 1.** (Continued)

| $N$ in $N$-fold | Rank of QC | Rotation angles | Special relations of rotation angles | Numbers of quadratic invariants | Quadratic invariants |
|---|---|---|---|---|---|
| 9 | 7 | $\alpha = \frac{2}{9}\pi$<br>$\beta = 2\alpha$<br>$\gamma = 4\alpha$ | $\alpha + 2\gamma = 2\pi$<br>$\gamma - \beta - 2\alpha = 0$<br>$\beta - 2\alpha = 0$ | $n_C = 5(5)$<br>$n_K = 10(8)$<br>$n_R = 4(2)$ | Essential phonon ($\vec{u}$) invariants $E_{ij}E_{kl}$;<br>essential phason ($\vec{w}$) invariants $W_{ij}W_{kl}$;<br>essential phason ($\vec{v}$) invariants $V_{ij}V_{kl}$:<br>$(V_{21} + V_{12})V_{13} + (V_{11} - V_{22})V_{23}$,<br>$\langle (V_{11} - V_{22})V_{13} - (V_{12} + V_{21})V_{23} \rangle$,<br>$(W_{11} - W_{22})(V_{11} + V_{22}) + (W_{12} + W_{21})(V_{21} - V_{12})$,<br>$\langle (W_{11} - W_{22})(V_{21} - V_{12}) - (W_{12} + W_{21})(V_{11} + V_{22}) \rangle$,<br>$(W_{21} - W_{12})E_{13} - (W_{11} + W_{22})E_{23}$,<br>$\langle E_{13}(W_{11} + W_{22}) + (W_{21} - W_{12})E_{23} \rangle$,<br>$(E_{11} - E_{22})W_{13} + 2E_{12}W_{23}$,<br>$\langle (E_{11} - E_{22})W_{23} - 2E_{12}W_{13} \rangle$, |
| 10 | 5 | $\alpha = \frac{1}{5}\pi$<br>$\beta = 3\alpha$ | $\beta - 3\alpha = 0$ | $n_C = 5(5)$<br>$n_K = 3(3)$<br>$n_R = 2(1)$ | Essential phonon invariants $E_{ij}E_{kl}$;<br>essential phason invariants $W_{ij}W_{kl}$:<br>$2(W_{21} - W_{12})E_{12} + (W_{11} + W_{22})(E_{11} - E_{22})$,<br>$\langle (E_{11} - E_{22})(W_{21} - W_{12}) - 2E_{12}(W_{11} + W_{22}) \rangle$ |
| 12 | 5 | $\alpha = \frac{1}{6}\pi$<br>$\beta = 5\alpha$ | $\beta + \alpha = \pi$ | $n_C = 5(5)$<br>$n_K = 5(4)$<br>$n_R = 0(0)$ | Essential phonon invariants $E_{ij}E_{kl}$;<br>essential phason invariants $W_{ij}W_{kl}$:<br>$(W_{11} - W_{22})^2$, $\langle (W_{11} - W_{22})(W_{21} + W_{12}) \rangle$ |
| 14 | 7 | $\alpha = \frac{1}{7}\pi$<br>$\beta = 3\alpha$<br>$\gamma = 5\alpha$ | $\gamma - \beta - 2\alpha = 0$<br>$\beta - 3\alpha = 0$ | $n_C = 5(5)$<br>$n_K = 8(6)$<br>$n_R = 2(1)$ | Essential phonon ($\vec{u}$) invariants $E_{ij}E_{kl}$;<br>essential phason ($\vec{w}$) invariants $W_{ij}W_{kl}$;<br>essential phason ($\vec{v}$) invariants $V_{ij}V_{kl}$:<br>$(W_{11} - W_{22})(V_{11} + V_{22}) + (W_{12} + W_{21})(V_{21} - V_{12})$,<br>$\langle (W_{11} - W_{22})(V_{21} - V_{12}) - (W_{12} + W_{21})(V_{11} + V_{22}) \rangle$,<br>$2(W_{21} - W_{12})E_{12} + (W_{11} + W_{22})(E_{11} - E_{22})$,<br>$\langle (E_{11} - E_{22})(W_{21} - W_{12}) - 2E_{12}(W_{11} + W_{22}) \rangle$ |
| 18 | 7 | $\alpha = \frac{1}{9}\pi$<br>$\beta = 5\alpha$<br>$\gamma = 7\alpha$ | $\gamma - \beta - 2\alpha = 0$ | $n_C = 5(5)$<br>$n_K = 8(6)$<br>$n_R = 0(0)$ | Essential phonon ($\vec{u}$) invariants $E_{ij}E_{kl}$;<br>essential phason ($\vec{w}$) invariants $W_{ij}W_{kl}$;<br>essential phason ($\vec{v}$) invariants $V_{ij}V_{kl}$:<br>$(W_{11} - W_{22})(V_{11} + V_{22}) + (W_{12} + W_{21})(V_{21} - V_{12})$,<br>$\langle (W_{11} - W_{22})(V_{21} - V_{12}) - (W_{12} + W_{21})(V_{11} + V_{22}) \rangle$ |

In (9), $\alpha = \frac{2}{5}\pi$, $\beta = 3\alpha$, $\alpha - 2\beta = -2\pi$; so, $[\partial_3 w_1, \partial_3 w_2]\begin{bmatrix}\partial_1 w_2 - \partial_2 w_1 \\ \partial_1 w_1 + \partial_2 w_2\end{bmatrix}$ transforms under the unit matrix and, besides the three essential phason invariants, another two quadratic invariants of phason strain are obtained as $\partial_3 w_1(\partial_1 w_2 - \partial_2 w_1) + \partial_3 w_2(\partial_1 w_1 + \partial_2 w_2)$ and $\partial_3 w_1(\partial_1 w_1 + \partial_2 w_2) - \partial_3 w_2(\partial_1 w_2 - \partial_2 w_1)$.

In (11), $\beta + 2\alpha = 2\pi$, $\beta - 3\alpha = 0$; so six invariants are obtained: $E_{13}(\partial_1 w_2 + \partial_2 w_1) + E_{23}(\partial_1 w_1 - \partial_2 w_2)$, $E_{13}(\partial_1 w_1 - \partial_2 w_2) - E_{23}(\partial_1 w_2 + \partial_2 w_1)$, $(E_{11} - E_{22})\partial_3 w_2 + 2E_{12}\partial_3 w_1$, $(E_{11} - E_{22})\partial_3 w_1 - 2E_{12}\partial_3 w_2$, $(E_{11} - E_{22})(\partial_1 w_1 + \partial_2 w_2) + 2E_{12}(\partial_1 w_2 - \partial_2 w_1)$ and $(E_{11} - E_{22})(\partial_1 w_2 - \partial_2 w_1) - 2E_{12}(\partial_1 w_1 + \partial_2 w_2)$.

### 3.3. $N = 7$: *the case of a two-dimensional quasicrystal with non-crystallographic symmetry and of rank 7*

Here, $\alpha = \frac{2}{7}\pi$, $\beta = 5\alpha$, $\gamma = 3\alpha$; the quadratic invariants of phonon strain take the same forms as $N = 5$. The quadratic invariants of phason strain are

(i) three essential self-product phason-invariants $\partial_j w_i\ \partial_l w_k$;

(ii) three essential self-product phason invariants $\partial_j v_i \partial_l v_k$, *plus* $\partial_3 v_1(\partial_1 v_2 + \partial_2 v_1) + \partial_3 v_2(\partial_1 v_1 - \partial_2 v_2)$ and $\partial_3 v_1(\partial_1 v_1 - \partial_2 v_2) - \partial_3 v_2(\partial_1 v_2 + \partial_2 v_1)$, where the latter two are due to $\alpha + 2\gamma = 2\pi$ in (9);

(iii) six cross-terms $\partial_j w_i\ \partial_l v_k$, namely $(\partial_1 w_1 + \partial_2 w_2)(\partial_1 v_1 - \partial_2 v_2) + (\partial_1 w_2 - \partial_2 w_1)(\partial_1 v_2 + \partial_2 v_1)$, $(\partial_1 w_1 + \partial_2 w_2)(\partial_1 v_2 + \partial_2 v_1) - (\partial_1 w_2 - \partial_2 w_1)(\partial_1 v_1 - \partial_2 v_2)$, and $\partial_3 w_1(\partial_1 v_2 - \partial_2 v_1) + \partial_3 w_2(\partial_1 v_1 + \partial_2 v_2)$, $\partial_3 w_1(\partial_1 v_1 + \partial_2 v_2) - \partial_3 w_2(\partial_1 v_2 - \partial_2 v_1)$, $(\partial_1 w_1 + \partial_2 w_2)\partial_3 v_2 + (\partial_1 w_2 - \partial_2 w_1)\partial_3 v_1$ $(\partial_1 w_1 + \partial_2 w_2)\partial_3 v_1 - (\partial_1 w_2 - \partial_2 w_1)\partial_3 v_2$ due to $2\alpha - \beta + \gamma = 0$ *and* $\gamma + \beta - \alpha = 2\pi$ in equation (10), respectively.

In (11), $\beta + 2\alpha = 2\pi$, $\gamma - 3\alpha = 0$; so there are six coupling invariants between phonon strain and phason strain: $E_{13}(\partial_1 w_2 + \partial_2 w_1) + E_{23}(\partial_1 w_1 - \partial_2 w_2)$, $E_{13}(\partial_1 w_1 - \partial_2 w_2) - E_{23}(\partial_1 w_2 + \partial_2 w_1)$, $(E_{11} - E_{22})\partial_3 w_2 + 2E_{12}\partial_3 w_1$, $(E_{11} - E_{22})\partial_3 w_1 - 2E_{12}\partial_3 w_2$, $(E_{11} - E_{22})(\partial_1 v_1 + \partial_2 v_2) + 2E_{12}(\partial_1 v_2 - \partial_2 v_1)$ and $(E_{11} - E_{22})(\partial_1 v_2 - \partial_2 v_1) - 2E_{12}(\partial_1 v_1 + \partial_2 v_2)$.

For $N$ equal to the other integers, one can use a similar procedure; all the results are listed in table 1.

According to conventions in crystallography [11], point groups which would become identical when a centre of symmetry is added belong to the same Laue class. It is obvious that all the phonon strains and phason strains are centrosymmetrical, i.e. that, under the action of the symmetry operation 'inversion', they remain unchanged. Therefore, elastic properties possess an intrinsic centrosymmetry, and hence all point groups belonging to the same Laue class have the same elastic properties. In the above, we discussed only the cases with $c_n$ symmetries. If we add $i$, $m_h$, $2_h$ or $m_v$ operations on the structure, some new symmetries are obtained and we divided all the quasicrystalline point groups into two types: type I and type II. When $N$ is odd, then point groups $N$ and $\bar{N}$ belong to type I with Abelian groups, $N2$, $Nm$ and $\bar{N}m$ belong to type II with non-Abelian groups; when $N$ is even, $N$, $\bar{N}$ and $N/m$ belong to type I, while $N22$, $Nmm$, $\bar{N}m2$ and $N/mmm$ belong to type II.

In table 1, we list all the quadratic invariants for $N = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18$. In the sixth column, for $N = 1, 2, 3, 4, 6$, A is a set of linear invariants for both type I and type II Laue classes, and B is another set of linear invariants for type I Laue classes, and the set of 1D antisymmetry basis vectors for type II Laue classes. So, the dot product of any two taken from sets A and B (containing a self-product) is a quadratic invariant for type I Laue classes, and the dot product of any two taken from the set A and that from the set B, except one from A and one from B are invariants for type II. We choose $m_v$ perpendicular to the $x$ axis or $2_h$ along the $x$ axis in the second Laue class in table 1. All the invariances in the sixth column hold for the type I Laue classes, and those in angular brackets hold for type II Laue classes. In the fifth column of table 1, $n_C$, $n_K$, $n_R$ are the numbers of quadratic invariants of phonon strain, phason strain and coupling between phonon strain and phason strain, respectively, and the numbers without parentheses and those in parentheses correspond to those of the type I and type II Laue classes, respectively.

## 4. Concluding remarks and discussion

We have demonstrated a method to derive the quadratic elastic invariants for all the 2D QCs of rank 5 and rank 7. The explicit forms are given for the 2D QCs with onefold, twofold, threefold, fourfold, fivefold, sixfold, sevenfold, eightfold, ninefold, tenfold, twelvefold, fourteenfold or eighteenfold rotational symmetry in table 1. From the results, one can see that, among these invariants, five quadratic invariants of phonon strain and three quadratic invariants of phason strain are essential for any $N$.

If one considers only the planar QCs, i.e. all the terms with subscript 3 are omitted, there are two quadratic invariants of phonon strain and two quadratic invariants of phason strain remaining; they are $(E_{11}+E_{22})^2$ and $(E_{11}-E_{22})^2+4E_{12}^2$, which are equivalent to $(\nabla\cdot u)^2$ and $E_{ij}E_{ij}(i, j = 1, 2)$, $(w_{11}-w_{22})^2+(w_{12}+w_{21})^2$ and $(w_{11}+w_{22})^2+(w_{12}-w_{21})^2$, which are equivalent to $w_{ij}w_{ij}$ and $s_{ij}w_{ij}w_{ji}$ where here

$$s_{ij} = \begin{cases} 1 & \text{for } i = j \\ -1 & \text{for } i \neq j. \end{cases}$$

For a conventional crystal, if there are only two quadratic elastic invariants (i.e. two independent elastic constants) in the basal plane, one can call it a transverse isotopic crystal. For a 2D QC, we can similarly define such a structure in whose quasiperiodic plane there are two quadratic invariants of phonon strain, namely $(\nabla\cdot u)^2$ and $E_{ij}E_{ij}$, and two quadratic invariants of phason strain namely $w_{ij}w_{ij}$ and $s_{ij}w_{ij}w_{ji}$, as a transverse isotopic 2D QC. Of course, it is not necessary to have coupling between the phonon strain and phason strain for any QC. However, the coupling between the phonon strain and phason strain may effect the elastic behaviour of the QC. For the planar cases, there is no coupling between the phonon strain and phason strain for $N = 9, 12, 18$.

If one considers the 2D QC of rank 9, $N = 15, 16, 20, 24$ and 30 are allowable. In these cases, there are three types of phason strain; a similar procedure can be used to determine their properties.

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